Second order perturbations of a Schwarzschild black hole: inclusion of odd parity perturbations

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We consider perturbations of a Schwarzschild black hole that can be of both even and odd parity, keeping terms up to second order in perturbation theory, for the $\ell = 2$ axisymmetric case. We develop explicit formulae for the evolution equations and radiated energies and waveforms using the Regge–Wheeler–Zerilli approach. This formulation is useful, for instance, for the treatment in the "close limit approximation" of the collision of counterrotating black holes.

I. INTRODUCTION

There has been a recent surge of interest in black hole perturbation theory, stemming from the successful applicability of such a formalism in the case of black hole collisions (see [1] for a review). In particular, the use of second order perturbative computations [2] has proven quite useful to endow the perturbative formalism with "error bars" and also to increase its accuracy.

The computations associated with the second order formulation are quite laborious. In particular, the second order perturbative equations are linear differential equations that contain "source" terms that are quadratic in the first order perturbations. This forbids us from doing a "generic" second order computation, since it would involve an infinite number of source terms. Computations have to be restricted to certain multipolar orders and parities to achieve a computable expression. The expressions we have presented in the past [2] are all for even parity axisymmetric $\ell = 2$ first order perturbations. This is a significant case, since for first order perturbations such modes are the ones that dominate the initial data for black hole collisions (at least in the head-on case, in the non-head on case one also has non-axisymmetric $m = \pm 2$ modes). If the black holes individually have spin, however, odd parity terms appear as well. If one wishes to study such collisions with the second order formulation one needs to lay out the second order equations for odd and even parity perturbations. Since the "source" terms that appear in the second order equations are quadratic, one has quadratic contributions from odd parity first order modes in the even parity second order equations. The intention of this paper is to present a comprehensive account of the perturbative equations involving odd terms, up to second order. In a separate publication we will apply the formalism to the collision of counterrotating black holes.

II. PERTURBATIVE FORMALISM: REVIEW OF SOME FIRST ORDER RESULTS

A. The Regge–Wheeler decomposition and the Regge–Wheeler gauge

In order to fix notation and to set the computational philosophy for the rest of the paper, we recollect here several results of first order perturbation theory that are known, but that it is useful to have laid out in a compact fashion for even and odd perturbations together.

We consider the axially symmetric perturbations of a spherically symmetric background, up to and including second order, using the Regge–Wheeler framework. In first order perturbations there is no loss of generality in the consideration of axisymmetric perturbations since modes with different m value decouple. This is not the case, however, in second order theory and our treatment is therefore restricted to the axisymmetric case. In terms of the usual Schwarzschild-like coordinates t, r, θ, ϕ, any $\ell = 2$ axially symmetric perturbation may be written in the "Regge–Wheeler form" [3]

$$\widetilde{g}_{tt} = -(1-2M/r)\left\{1 - \left[\epsilon \widetilde{H}_0^{(1)}(r,t) + \epsilon^2 \widetilde{H}_0^{(2)}(r,t)\right] P_2(\theta)\right\} \tag{1}$$

$$\widetilde{g}_{rr} = (1-2M/r)^{-1}\left\{1 + \left[\epsilon \widetilde{H}_2^{(1)}(r,t) + \epsilon^2 \widetilde{H}_2^{(2)}(r,t)\right] P_2(\theta)\right\} \tag{2}$$

$$\widetilde{g}_{rt} = \left[\epsilon \widetilde{H}_1^{(1)}(r,t) + \epsilon^2 \widetilde{H}_1^{(2)}(r,t)\right] P_2(\theta) \tag{3}$$

$$\widetilde{g}_{t\theta} = \left[\epsilon \widetilde{h}_0^{(1)}(r,t) + \epsilon^2 \widetilde{h}_0^{(2)}(r,t)\right] \partial P_2(\theta)/\partial\theta) \tag{4}$$

$$\widetilde{g}_{t\phi} = \left[\epsilon \widetilde{h}_{0o}^{(1)}(r,t) + \epsilon^2 \widetilde{h}_{0o}^{(2)}(r,t)\right] \sin(\theta)\partial P_2(\theta)/\partial\theta) \tag{5}$$

$$\widetilde{g}_{r\phi} = \left[\epsilon \widetilde{h}_{1o}^{(1)}(r,t) + \epsilon^2 \widetilde{h}_{1o}^{(2)}(r,t)\right] \sin(\theta)\partial P_2(\theta)/\partial\theta) \tag{6}$$

$$\widetilde{g}_{r\theta} = \left[\epsilon \widetilde{h}_1^{(1)}(r,t) + \epsilon^2 \widetilde{h}_1^{(2)}(r,t)\right] \partial P_2(\theta)/\partial\theta) \tag{7}$$

$$\begin{aligned}\widetilde{g}_{\theta\theta} = r^2 \Big\{ & 1 + [\epsilon \widetilde{K}^{(1)}(r,t) + \epsilon^2 \widetilde{K}^{(2)}(r,t)] P_2(\theta) \\ & + [\epsilon \widetilde{G}^{(1)}(r,t) + \epsilon^2 \widetilde{G}^{(2)}(r,t)] \partial^2 P_2(\theta)/\partial\theta^2) \Big\}\end{aligned} \tag{8}$$

$$\begin{aligned}\widetilde{g}_{\phi\phi} = r^2 \Big\{ & \sin^2\theta + \sin^2\theta [\epsilon \widetilde{K}^{(1)}(r,t) + \epsilon^2 \widetilde{K}^{(2)}(r,t)] P_2(\theta) \\ & + \sin(\theta)\cos(\theta)[\epsilon \widetilde{G}^{(1)}(r,t) + \epsilon^2 \widetilde{G}^{(2)}(r,t)] \partial P_2(\theta)/\partial\theta) \Big\}\end{aligned} \tag{9}$$

$$\widetilde{g}_{\theta\phi} = \frac{1}{2}\left[\epsilon \widetilde{h}_{2o}^{(1)}(r,t) + \epsilon^2 \widetilde{h}_{2o}^{(2)}(r,t)\right] \left[\cos(\theta)\partial P_2(\theta)/\partial\theta - \sin(\theta)\partial^2 P_2(\theta)/\partial\theta^2\right] \tag{10}$$

where $P_2(\theta) = 3\cos^2(\theta)/2 - 1/2$. We introduce the expansion parameter ϵ, which defines the perturbation order, and identify the corresponding metric coefficients with the superscripts (1), (2) and use the subscript "o" to refer to the odd parity quantities.

As we stated above, any axisymmetric perturbation can be put into the Regge–Wheeler form. However, for calculational simplicity, it is usually convenient to take advantage of the coordinate freedom to restrict somewhat the form of the metric. An example of such choice is the so called *Regge - Wheeler gauge* (RWG), where the nonvanishing metric coefficients are

$$g_{tt} = -(1-2M/r)\left\{1 - [\epsilon H_0^{(1)}(r,t) + \epsilon^2 H_0^{(2)}(r,t)] P_2(\theta)\right\} \tag{11}$$

$$g_{rr} = (1-2M/r)^{-1}\left\{1 + [\epsilon H_2^{(1)}(r,t) + \epsilon^2 H_2^{(2)}(r,t)] P_2(\theta)\right\} \tag{12}$$

$$g_{rt} = [\epsilon H_1^{(1)}(r,t) + \epsilon^2 H_1^{(2)}(r,t)] P_2(\theta) \tag{13}$$

$$g_{t\theta} = [\epsilon h_{0o}^{(1)}(r,t) + \epsilon^2 h_{0o}^{(2)}(r,t)] \sin(\theta)\partial P_2(\theta)/\partial\theta \tag{14}$$

$$g_{r\theta} = [\epsilon h_{1o}^{(1)}(r,t) + \epsilon^2 h_{1o}^{(2)}(r,t)] \sin(\theta)\partial P_2(\theta)/\partial\theta \tag{15}$$

$$g_{\theta\theta} = r^2\left\{1 + [\epsilon K^{(1)}(r,t) + \epsilon^2 K^{(2)}(r,t)] P_2(\theta)\right\} \tag{16}$$

$$g_{\phi\phi} = r^2\left\{\sin^2\theta + \sin^2\theta [\epsilon K^{(1)}(r,t) + \epsilon^2 K^{(2)}(r,t)] P_2(\theta)\right\}, \tag{17}$$

that is, we are demanding that $h_0^{(i)} = h_1^{(i)} = G^{(i)} = h_{2o}^{(i)} = 0$, $i = 1, 2$.

A remarkable aspect of the Regge–Wheeler gauge (RWG) is that it is *unique*, in the sense that the metric perturbation coefficients in the RWG can be uniquely recovered from those in an arbitrary gauge. That is, given an arbitrary metric, in an arbitrary gauge, there is a well defined procedure that brings it to the RWG.

Let us exhibit this property for first order gauge transformations. One can uniquely choose a gauge transformation vector that brings the metric to the RWG form [2]. The resulting transformation formulae are,

$$K^{(1)} = \widetilde{K}^{(1)} + (r-2M)\left(\widetilde{G}^{(1)}{}_{,r} - \frac{2}{r^2}\widetilde{h}_1^{(1)}\right) \tag{18}$$

$$H_2^{(1)} = \widetilde{H}_2^{(1)} + (2r-3M)\left(\widetilde{G}^{(1)}{}_{,r} - \frac{2}{r^2}\widetilde{h}_1^{(1)}\right) + r(r-2M)\left(\widetilde{G}^{(1)}{}_{,r} - \frac{2}{r^2}\widetilde{h}_1^{(1)}\right)_{,r} \tag{19}$$

$$H_1^{(1)} = \widetilde{H}_1^{(1)} + r^2\widetilde{G}^{(1)}{}_{,tr} - \widetilde{h}_1^{(1)}{}_{,t} - \frac{2M}{r(r-2M)}\widetilde{h}_0^{(1)} + \widetilde{h}_0^{(1)}{}_{,r} + \frac{r(r-3M)}{r-2M}\widetilde{G}^{(1)}{}_{,t} \tag{20}$$

$$H_0^{(1)} = \widetilde{H}_0^{(1)} - M\left(\widetilde{G}^{(1)}{}_{,r} - \frac{2}{r^2}\widetilde{h}_1^{(1)}\right) - \frac{2r}{r-2M}\widetilde{h}_0^{(1)}{}_{,t} + \frac{r^3}{(r-2M)}\widetilde{G}^{(1)}{}_{,tt} \tag{21}$$

$$h_{0o}^{(1)} = \frac{1}{2}\widetilde{h}_{2o,t}^{(1)} + \widetilde{h}_{0o}^{(1)} \tag{22}$$

$$h_{1o}^{(1)} = \frac{1}{2r}\left(r\widetilde{h}_{2o,r}^{(1)} - 2\widetilde{h}_{2o}^{(1)}\right) + \widetilde{h}_{1o}^{(1)}. \tag{23}$$

Something that should be stressed is that given the uniqueness of the RWG, quantities computed in such a gauge (like, for instance, metric components), are *gauge invariant*. That is, substituting the above formulae in the definition of any quantity in the RWG one obtains expressions for the quantity in any gauge.

The procedure can in principle be repeated at every order in perturbation theory. To achieve this, one first performs a first order transformation that yields the metric (at first order) in the RWG. This transformation will induce transformations at all higher orders as well. One then makes a *purely second order* gauge transformation that brings the second order portion of the metric to the RWG. This will leave the first order piece in RWG, and will modify the third and higher orders as well. One can continue this procedure up to an arbitrarily high order and the metric will be, up to that order, in the RWG. Unfortunately, the procedure is not quite unique at higher orders. The reason for this is that if before performing the second order gauge transformation, one performs an $\ell = 0$ first order gauge transformation, the procedure we outlined yields at the end of the day a *different* second order metric, but *still in the RWG*. That is, the metric one obtains at the end of the procedure is not unique, in spite of being in the RWG. Fortunately, one can isolate second order quantities that are invariant under these $\ell = 0$ transformations. This point has been discussed in reference [4] in detail, so we will not repeat it here. From the expressions for waveforms given in this paper, one can apply the techniques of reference [4] in order to compare with numerical results without the $\ell = 0$ ambiguity.

From now on, we will do computations in the RWG. Because of what we just discussed, there is no issue of gauge invariance in our formalism, since all quantities in the RWG (appropriately modified to take into account the $\ell = 0$ issue) will be "gauge invariant" in the sense described above.

B. The first order Zerilli equation

As shown by Zerilli, we may introduce a function $\psi^{(1)}(r,t)$, such that if we write for the metric components in the RWG,

$$K^{(1)}(r,t) = 6\frac{r^2+rM+M^2}{r^2(2r+3M)}\psi^{(1)}(r,t) + \left(1-2\frac{M}{r}\right)\frac{\partial\psi^{(1)}(r,t)}{\partial r} \tag{24}$$

$$H_0^{(1)}(r,t) = \frac{\partial}{\partial r}\left[\frac{2r^2-6rM-3M^2}{r(2r+3M)}\psi^{(1)}(r,t) + (r-2M)\frac{\partial\psi^{(1)}(r,t)}{\partial r}\right] - K^{(1)}(r,t) \tag{25}$$

$$H_1^{(1)}(r,t) = \frac{2r^2-6rM-3M^2}{(r-2M)(2r+3M)}\frac{\partial\psi^{(1)}(r,t)}{\partial t} + r\frac{\partial^2\psi^{(1)}(r,t)}{\partial r\partial t} \tag{26}$$

$$H_2^{(1)}(r,t) = H_0^{(1)}(r,t), \tag{27}$$

then the linearized Einstein equations are satisfied, provided only that $\psi^{(1)}(r,t)$ is a solution of the Zerilli equation (28).

$$\frac{\partial^2\psi^{(1)}(r,t)}{\partial r^{*2}} - \frac{\partial^2\psi^{(1)}(r,t)}{\partial t^2} - V_Z(r^*)\psi^{(1)}(r,t) = 0 \tag{28}$$

where

$$r^* = r + 2M\ln[r/(2M)-1] \tag{29}$$

and

$$V_Z(r) = 6\left(1-2\frac{M}{r}\right)\frac{4r^3+4r^2M+6rM^2+3M^3}{r^3(2r+3M)^2} \tag{30}$$

It is straightforward to obtain "inversion" formulas for $\psi^{(1)}(r,t)$ in terms of the metric coefficients. From (24) and (26), we have

$$\frac{\partial\psi^{(1)}(r,t)}{\partial t} = \frac{r}{2r+3M}\left[r\frac{\partial K^{(1)}}{\partial t} - \left(1-2\frac{M}{r}\right)H_1^{(1)}\right] \tag{31}$$

while from (24) and (25) we find

$$\psi^{(1)}(r,t) = \frac{r(r-2M)}{3(2r+3M)}\left(H_0^{(1)} - r\frac{\partial K^{(1)}}{\partial r}\right) + \frac{r}{3}K^{(1)} \tag{32}$$

Equations (31) and (32) are equivalent, as far as first order perturbations are concerned, up to an additive function of r. This ambiguity is of no concern since the relevant physical quantities, like radiated energies and waveforms are all computed in terms of the time derivative of $\psi^{(1)}$, so it is conventionally ignored, we will assume we set the additive function to zero. Equation (32) can be used to obtain a gauge invariant form for $\psi^{(1)}(r,t)$, i.e., an expression for $\psi^{(1)}(r,t)$, valid in an *arbitrary gauge*. This is an especially important result, because it allows us to obtain the initial data for the Zerilli equation, directly in an appropriate gauge. It should be noted, however, that although $\psi^{(1)}(r,t)$ is uniquely defined in a general gauge by (32), we do not have, in a general gauge, unique expressions for the metric perturbation functions in terms of $\psi^{(1)}(r,t)$. The general statement of Zerilli's results is, in this case, that if the metric perturbations satisfy the first order Einstein equations, then $\psi^{(1)}(r,t)$ satisfies the Zerilli equation (28).

The situation, as we shall see, is somewhat different for the second order perturbations, because of the presence of "source terms" in the Zerilli equation.

C. The Odd parity equation

One can also write a "Zerilli"-like equation (known as the Regge–Wheeler equation, historically it was discovered earlier than the even parity one [3]) for the odd part of first order metric perturbations and the equations they satisfy. If we define

$$h_{0o}^{(1)}(r,t) = \left(1 - \frac{2M}{r}\right)\left(r\frac{\partial}{\partial r}Q^{(1)}(r,t) + Q^{(1)}(r,t)\right) \tag{33}$$

$$h_{1o}^{(1)}(r,t) = \frac{r}{1-2M/r}\frac{\partial}{\partial t}Q^{(1)}(r,t) \tag{34}$$

the Einstein equations are satisfied, provided that $Q^{(1)}(r,t)$ is a solution of the equation

$$\frac{\partial^2 Q^{(1)}(r,t)}{\partial r^{*2}} - \frac{\partial^2 Q^{(1)}(r,t)}{\partial t^2} - V_Q(r^*)Q^{(1)}(r,t) = 0 \tag{35}$$

where

$$r^* = r + 2M\ln[r/(2M) - 1] \tag{36}$$

and

$$V_Q(r) = 6\frac{(r-M)(r-2M)}{r^4}. \tag{37}$$

It is easier to obtain inversion formulas in this case that in the even parity case. We can write,

$$Q^{(1)}(r,t) = -\frac{1}{4}\left(r\frac{\partial h_{1o}^{(1)}(r,t)}{\partial t} - r\frac{\partial h_{0o}^{(1)}(r,t)}{\partial r} + 2h_{0o}^{(1)}(r,t)\right) \tag{38}$$

$$\frac{\partial Q^{(1)}(r,t)}{\partial t} = \frac{(1-2M/r)}{r}h_{1o}^{(1)}(r,t) \tag{39}$$

We can make the same comments as in the even case. Equation (38) can be used to give a gauge invariant form for the odd parity Zerilli function. Both equations allow us to give initial conditions for $Q^{(1)}(r,t)$ in an any gauge. As in the even case, there is no unique way to write the metric perturbations in terms of $Q^{(1)}(r,t)$ in an arbitrary gauge.

III. SECOND ORDER PERTURBATIONS

We define the *second order Regge–Wheeler gauge* by imposing that $h_0^{(i)} = h_1^{(i)} = G^{(i)} = h_{2o}^{(i)} = 0$, $i = 1, 2$. It is easy to check that this gauge is uniquely defined, as long as we restrict to $\ell = 2$ axisymmetric first and second order gauge transformations, but the general expressions for the second order gauge transformations are rather lengthy.

It should be clear from the perturbation expansion, that the second order Einstein equations imply, in turn, that the second order perturbation functions satisfy the same linear set of equations as the first order perturbations, but with additional terms quadratic in the first order perturbations, that may be thought of as representing “sources” for these equations.

Therefore if motivated by the corresponding expression for the time derivatives $\partial\psi^{(1)}/\partial t$, $\partial Q^{(1)}/\partial t$, of the Zerilli functions in first order perturbation theory, we introduce gauge invariant functions, given by

$$\chi^{(2)}(r,t) = \frac{r}{2r+3M}\left[r\frac{\partial K^{(2)}(r,t)}{\partial t} - \left(1-\frac{2M}{r}\right)H_1^{(2)}(r,t)\right] \tag{40}$$

$$\Theta^{(2)}(r,t) = \frac{(1-2M/r)}{r}h_{1o}^{(2)}(r,t) \tag{41}$$

it follows that if the second order Einstein equations are satisfied, then $\chi^{(2)}(r,t)$, $\Theta^{(2)}$ satisfy equations of the form

$$\frac{\partial^2\chi^{(2)}(r,t)}{\partial {r^*}^2} - \frac{\partial^2\chi^{(2)}(r,t)}{\partial t^2} - V_Z(r^*)\chi^{(2)}(r,t) + \mathcal{S}_Z = 0 \tag{42}$$

$$\frac{\partial^2\Theta^{(2)}(r,t)}{\partial {r^*}^2} - \frac{\partial^2\Theta^{(2)}(r,t)}{\partial t^2} - V_Q(r^*)\Theta^{(2)}(r,t) + \mathcal{S}_Q = 0 \tag{43}$$

where $\mathcal{S}_Z$, $\mathcal{S}_Q$ are quadratic polynomials in the first order functions and their t and r derivatives, with r dependent coefficients. $\mathcal{S}_Z$,$\mathcal{S}_Q$ may be considered as a kind of ”source terms” for the otherwise homogeneous equations satisfied by $\chi^{(2)}$, $\Theta^{(2)}$.

But, precisely because of the presence of these “source terms”, if we redefine $\chi^{(2)}(r,t)$, $\Theta^{(2)}(r,t)$ by the addition of a quadratic polynomial in the first order functions and their derivatives, the new, redefined $\chi^{(2)}(r,t)$, $\Theta^{(2)}(r,t)$ will still satisfy an equation of the form (42), provided only that the first and second order Einstein equations are satisfied.

This arbitrariness in the definition of the Zerilli functions associated to the second order perturbations may be used to simplify the form, and asymptotic behavior of the “source terms” $\mathcal{S}_Z$, $\mathcal{S}_Q$. As was discussed in [2], the source terms are delicate in the sense that they can be divergent for large values of t and r. The divergence of the source does not imply a physical divergence in the problem, but it poses serious difficulties for numeric computations. If we make the following choice, based on the study of the solutions of the Zerilli equations for large values of r, the source simplifies and also is well behaved at spatial infinity,

$$\chi^{(2)}(r,t) = \frac{r}{2r+3M}\left[r\frac{\partial K^{(2)}}{\partial t} - \left(1-\frac{2M}{r}\right)H_1^{(2)}\right] - \frac{2}{7}\left[\frac{r^2}{2r+3M}K^{(1)}\frac{\partial K^{(1)}}{\partial t} + (K^{(1)})^2\right] \tag{44}$$

as our definition for $\chi^{(2)}(r,t)$, while $\Theta^{(2)}(r,t)$ is defined by (41). Making the appropriate replacements, we then find for the sources,

$$\begin{aligned}
\mathcal{S}_Z = & -\frac{12}{7}\frac{(r-2M)^3}{r^3(2r+3M)}\left[\frac{\left(-24r^5-108Mr^4-72M^2r^3+24M^3r^2+180M^4r+180M^5\right)(\psi)^2}{r^5\,(2r+3M)\,(r-2M)^2}\right. \\
& + \frac{\left(64r^5+176Mr^4+592M^2r^3+1020M^3r^2+1122M^4r+540M^5\right)\psi_r\psi}{r^4\,(2r+3M)^2\,(r-2M)} \\
& - \frac{\left(112r^5+480Mr^4+692M^2r^3+762M^3r^2+441M^4r+144M^5\right)\psi_t\psi}{r^2\,(2r+3M)^3\,(r-2M)^2} \\
& + \frac{\left(8r^3+16r^2M+36rM^2+24M^3\right)\psi_{rr}\psi}{r^3\,(2r+3M)} + \frac{\left(8r^2+12rM+7M^2\right)\psi_{rt}\psi}{r\,(2r+3M)\,(-r+2M)} + \frac{M\psi\psi_{rrt}}{2r+3M} + \frac{r\psi_t\psi_{rrr}}{3} \\
& - \frac{\left(12r^3+36r^2M+59rM^2+90M^3\right)(\psi_r)^2}{3r^3\,(r-2M)} + \frac{\left(-18r^3+4r^2M+33rM^2+48M^3\right)\psi_r\psi_t}{3\,(2r+3M)\,r\,(r-2M)^2} \\
& - \frac{\left(12r^2+20rM+24M^2\right)\psi_r\psi_{rr}}{3r^2} - \frac{(-3r+7M)\,\psi_r\psi_{rt}}{-3r+6M} - \frac{r\psi_r\psi_{rrt}}{3} \\
& + \frac{12\left(r^2+rM+M^2\right)^2(\psi_t)^2}{(2r+3M)\,r\,(r-2M)^3} - \frac{\left(-2r^2+M^2\right)\psi_{rr}\psi_t}{(2r+3M)\,(r-2M)} + \frac{\left(4r^2+4rM+4M^2\right)\psi_t\psi_{rt}}{(r-2M)^2} \\
& - \frac{(2r+3M)\,(r-2M)\,(\psi_{rr})^2}{3r} - \frac{(2r+3M)\,r\,(\psi_{rt})^2}{-3r+6M} - \frac{(-2r+10M)\,(3r+8M)\,Q_rQ_t}{r\,(r-2M)^2}
\end{aligned}$$

$$
+ \frac{\left(22r^2+70rM+66M^2\right)Q_rQ_{rt}}{(2r+3M)\,(r-2M)} - \frac{\left(-136r^3-462r^2M+50rM^2+456M^3\right)QQ_t}{r^2\,(2r+3M)\,(r-2M)^2}
$$

$$
\left. + \frac{\left(14r^2+58rM+66M^2\right)QQ_{rt}}{r\,(2r+3M)\,(r-2M)} - \frac{(22r+54M)\,(r+M)\,Q_tQ_{rr}}{(2r+3M)\,(r-2M)} \right] \tag{45}
$$

$$
\begin{aligned}
\mathcal{S}_Q = &-\frac{12}{7}\frac{(r-2M)^3}{r^3(2r+3M)}\left[\frac{\left(40r^4-150Mr^3-138r^2M^2+162M^3r+81M^4\right)Q\psi_t}{3r^3\,(2r+3M)\,(r-2M)^2}\right.\\
&+\frac{(2r+3M)\,(-r+M)\,Q\psi_{rrt}}{r\,(r-2M)}+\frac{\left(4r^3-17r^2M-33rM^2+81M^3\right)Q\psi_{rt}}{3r^2\,(r-2M)^2}\\
&+\frac{\left(-160M^2r^3+216M^5+102M^3r^2-160Mr^4+387M^4r-128r^5\right)Q_t\psi}{2r^3\,(2r+3M)^2\,(r-2M)^2}\\
&+\frac{\left(75M^3r^2-12r^5+81M^5-10M^2r^3+14Mr^4+150M^4r\right)\psi Q_{rt}}{r^2\,(2r+3M)^2\,(r-2M)^2}-\frac{(2r+3M)\,r\psi_{rr}Q_{rrt}}{3}\\
&+\frac{\left(4r^3+4r^2M+6rM^2+3M^3\right)\psi Q_{rrt}}{r\,(2r+3M)\,(r-2M)}-\frac{\left(-8r^4+56Mr^3-40r^2M^2-39M^3r+72M^4\right)Q_t\psi_r}{2r^2\,(2r+3M)\,(-r+2M)^2}\\
&-\frac{\left(-18r^4+19Mr^3-13r^2M^2+78M^3r+81M^4\right)\psi_rQ_{rt}}{3r\,(2r+3M)\,(r-2M)^2}+\frac{\left(2r^2-2rM+3M^2\right)\psi_rQ_{rrt}}{-3r+6M}\\
&-\frac{\left(28r^4-72Mr^3-96r^2M^2+114M^3r+81M^4\right)\psi_tQ_r}{3r^2\,(2r+3M)\,(r-2M)^2}+\frac{\left(-2r^2+6rM+3M^2\right)\psi_tQ_{rrr}}{-3r+6M}\\
&-\frac{\left(2r^4-9Mr^3+9r^2M^2+22M^3r+9M^4\right)\psi_tQ_{rr}}{r\,(2r+3M)\,(r-2M)^2}+\frac{\left(4r^3-11r^2M-15rM^2+63M^3\right)Q_t\psi_{rr}}{3r\,(r-2M)^2}\\
&+\frac{\left(6r^2+7rM+18M^2\right)\psi_{rr}Q_{rt}}{-6r+12M}-\frac{\left(10r^3-10r^2M-37rM^2+27M^3\right)\psi_{rt}Q_r}{3r\,(r-2M)^2}\\
&-\frac{\left(6r^2+7rM+18M^2\right)\psi_{rt}Q_{rr}}{-6r+12M}+\frac{(2r+3M)\,rQ_{rrr}\psi_{rt}}{3}-\frac{(2r+3M)\,(-r+3M)\,Q_t\psi_{rrr}}{-3r+6M}\\
&\left.-\frac{(2r+3M)\,r\psi_{rrr}Q_{rt}}{3}+\frac{(2r+3M)\,(-r+M)\,\psi_{rrt}Q_r}{-3r+6M}+\frac{(2r+3M)\,rQ_{rr}\psi_{rrt}}{3}\right]
\end{aligned} \tag{46}
$$

where $\psi = \psi^{(1)}$, $Q = Q^{(1)}$ and the subindices indicate partial derivatives.

IV. GRAVITATIONAL WAVE AMPLITUDE

A straightforward computational way to define a gravitational wave amplitude, is to consider a gauge in which the space-time is manifestly asymptotically flat. For instance, consider the typical gauge discussed in reference [5]. In this gauge the metric has components that behave in the following way,

$$
\begin{aligned}
H_2 &\simeq O(1/r^3)\\
h_1 &\simeq O(1/r)\\
K &\simeq O(1/r)\\
G &\simeq O(1/r)\\
h_{1o} &\simeq O(1/r)\\
h_{2o} &\simeq O(r)
\end{aligned} \tag{47}
$$

and, of course, $H_0 = H_1 = h_0 = h_{0o} = 0$. The gravitational wave amplitude can then simply be read off from the components K, G and h_{2o}. We therefore need to relate the behavior of these components of the metric to the Zerilli function in an asymptotic region far away from the source.

In order to study the asymptotic behavior, one considers a solution of the Zerilli equation in terms of an expansion in inverse powers of r with coefficients that are functions of the retarded time $t - r^*$, as discussed in detail in [2]. Substituting in the Zerilli equations we obtain,

$$
\begin{aligned}
\psi^{(1)} &= F_\psi^{(iii)}(t-r^*) + \frac{3}{r}F_\psi^{(ii)}(t-r^*) - \frac{3}{4r^2}\left[-7F_\psi^{(ii)}(t-r^*)M + 4F_\psi^{(i)}(t-r^*)\right] + O(1/r^3) \\
Q^{(1)} &= F_Q^{(iii)}(t-r^*) + \frac{3}{r}F_Q^{(ii)}(t-r^*) \\
&\quad + \frac{1}{r^2}\left[-\frac{3}{2}MF_Q^{(ii)}(t-r^*) + 3F_Q^{(i)}(t-r^*)\right] + O(1/r^3)
\end{aligned}
\tag{48}
$$

For this behavior of first order Zerilli functions, the sources S_Z, S_Q for second order Zerilli functions $\chi^{(2)}$, $\Theta^{(2)}$ decay as $O(1/r^2)$ asymptotically. Therefore, $\chi^{(2)}$,$\Theta^{(2)}$ will behave in a similar way to $\psi^{(1)}$, $Q^{(1)}$ for large r:

$$
\begin{aligned}
\chi^{(2)} &= F_\chi^{(20)}(t-r^*) + \frac{1}{r}F_\chi^{(21)}(t-r^*) + O(1/r^2) \\
\Theta^{(2)} &= F_Q^{(20)}(t-r^*) + \frac{1}{r}F_Q^{(21)}(t-r^*) + O(1/r^2)
\end{aligned}
\tag{49}
$$

We can now compute the asymptotic form for the metric perturbations, to first order and second order, in Regge–Wheeler gauge, using (48) and (49) in (25) and (27). Unfortunately, in the Regge–Wheeler gauge the metric is not in the manifestly asymptotically flat form (47). We therefore need to perform a gauge transformation to bring it to such a form, to first and second order.

In order to do this, we start by recalling that for any $\ell = 2$ axisymmetric gauge transformation, we can write the gauge transformation vectors as,

$$
\begin{array}{lcllcl}
\xi^{(1)t}_{(even)} &=& \mathcal{M}_0^{(1)}(r,t)P_2(\cos(\theta)) & \xi^{(1)r}_{(even)} &=& \mathcal{M}_1^{(1)}(r,t)P_2(\cos(\theta)) \\
\xi^{(1)\theta}_{(even)} &=& \mathcal{M}^{(1)}(r,t)\partial P_2(\cos(\theta))/\partial\theta & \xi^{(1)\phi}_{(even)} &=& 0
\end{array}
\tag{50}
$$

for even parity, and

$$
\begin{array}{lcllcl}
\xi^{(1)t}_{(odd)} &=& 0 & \xi^{(1)r}_{(odd)} &=& 0 \\
\xi^{(1)\theta}_{(odd)} &=& 0 & \xi^{(1)\phi}_{(odd)} &=& \mathcal{M}_2^{(1)}(r,t)\sin(\theta)^{-1}\partial P_2(\cos(\theta))/\partial\theta
\end{array}
\tag{51}
$$

for the odd parity case.

To achieve the asymptotically flat gauge (47) we demand that $H_0^{(1)}(r,t) = H_1^{(1)}(r,t) = h_0^{(1)}(r,t) = h_{0o}^{(1)} = 0$. From the general form of the gauge transformation equations we have,

$$
\begin{aligned}
0 &= \widetilde{H}_0^{(1)} + \frac{2M}{r(r-2M)}\mathcal{M}_1^{(1)} + 2\frac{\partial\mathcal{M}_0^{(1)}}{\partial t} \\
0 &= \widetilde{H}_1^{(1)} - \frac{r}{r-2M}\frac{\partial\mathcal{M}_1^{(1)}}{\partial t} + \frac{r-2M}{r}\frac{\partial\mathcal{M}_0^{(1)}}{\partial r} \\
0 &= -r^2\frac{\partial\mathcal{M}^{(1)}}{\partial t} + \frac{r-2M}{r}\mathcal{M}_0^{(1)} \\
H_2^{(1)} &= \widetilde{H}_2^{(1)} + \frac{2M}{r(r-2M)}\mathcal{M}_1^{(1)} - 2\frac{\partial\mathcal{M}_1^{(1)}}{\partial r} \\
h_1^{(1)} &= -\frac{r}{r-2M}\mathcal{M}_1^{(1)} - r^2\frac{\partial\mathcal{M}^{(1)}}{\partial r} \\
G^{(1)} &= -2\mathcal{M}^{(1)} \\
K^{(1)} &= \widetilde{K}^{(1)} - \frac{2}{r}\mathcal{M}_1^{(1)} \\
0 &= \widetilde{h}_{0o}^{(1)} + r^2\frac{\partial\mathcal{M}_2^{(1)}(r,t)}{\partial t} \\
h_{1o}^{(1)} &= \widetilde{h}_{1o}^{(1)} + r^2\frac{\partial\mathcal{M}_2^{(1)}(r,t)}{\partial r} \\
h_{2o}^{(1)} &= -2r^2\mathcal{M}_2^{(1)}(r,t)
\end{aligned}
\tag{52}
$$

where the quantities with tildes are given in the Regge–Wheeler gauge, $\{\mathcal{M}^{(1)}, \mathcal{M}_0^{(1)}, \mathcal{M}_1^{(1)}\}$, are the components of the even parity gauge transformation vector, and $\mathcal{M}_2^{(1)}$ that of the odd parity gauge transformation vector.

We can see that, given $\widetilde{H}_0^{(1)}$ and $\widetilde{H}_1^{(1)}$, the first and second equation above may be used to solve for $\mathcal{M}_0^{(1)}$ and $\mathcal{M}_1^{(1)}$. Then, the third equation leads to a partial differential equation for $\mathcal{M}^{(1)}$ while the equation for $h_{0o}^{(1)}$ fixes $\mathcal{M}_2^{(1)}$. Therefore, the set of equations, and therefore the gauge choice, is consistent. The solution, assuming only that $\widetilde{H}_0^{(1)}$, $\widetilde{H}_1^{(1)}$, $\widetilde{H}_2^{(1)}$, and $\widetilde{K}^{(1)}$ are given, is non unique since one has to integrate partial differential equations, so the solution in general involves free functions. One may choose such functions in a judicious way to ensure that the non-vanishing metric components behave in a manner consistent with asymptotic flatness.

Similarly, we define the second order asymptotically flat gauge by imposing the first order asymptotically flat gauge conditions and $H_0^{(2)}(r,t) = 0$, $H_1^{(2)}(r,t) = 0$, $h_0^{(2)}(r,t) = 0$ and $h_{0o}^{(2)}(r,t) = 0$. Clearly, the same reasoning as above, applied to the second order gauge transformations, shows that this is a consistent choice.

The end result of these calculations is that one may write the first order gauge transformation vectors in the form,

$$\begin{aligned}
\mathcal{M}^{(1)}(r,t) &= \frac{1}{2r}F_\psi^{(iii)}(t-r^*) + \frac{1}{r^2}F_\psi^{(ii)}(t-r^*) + \frac{1}{r^3}\left[\frac{1}{4}MF_\psi^{(ii)}(t-r^*) + \frac{3}{2}F_\psi^{(i)}(t-r^*)\right] + O(1/r^4) \\
\mathcal{M}_0^{(1)}(r,t) &= \frac{1}{2}rF_\psi^{(iv)}(t-r^*) + \left[F_\psi^{(iii)}(t-r^*) + MF_\psi^{(iv)}(t-r^*)\right] \\
&\quad + \frac{1}{r}\left[\frac{9}{4}MF_\psi^{(iii)}(t-r^*) + \frac{3}{2}F_\psi^{(ii)}(t-r^*) + 2M^2F_\psi^{(iv)}(t-r^*)\right] + O(1/r^2) \\
\mathcal{M}_1^{(1)}(r,t) &= \frac{1}{2}rF_\psi^{(iv)}(t-r^*) + \frac{3}{2}F_\psi^{(iii)}(t-r^*) \\
&\quad + \frac{1}{r}\left[\frac{3}{2}F_\psi^{(ii)}(t-r^*) - \frac{3}{4}MF_\psi^{(iii)}(t-r^*)\right] + O(1/r^2) \\
\mathcal{M}_2^{(1)}(r,t) &= \frac{3}{r}F_Q^{(iii)}(t-r^*) + \frac{2}{r^2}F_Q^{(ii)}(t-r^*) \\
&\quad + \frac{1}{r^3}\left[3F_Q^{(i)}(t-r^*) + \frac{1}{2}MF_Q^{(ii)}(t-r^*)\right] + O(1/r^4)
\end{aligned} \tag{53}$$

The second order gauge equations are more complex then the first order ones. It can be seen that the equations can also be solved iteratively for all orders in r, but for simplicity we only compute the terms that are going to be relevant in the gravitational waveforms. The meaningful terms are given by

$$\begin{aligned}
\mathcal{M}^{(2)} &= \frac{1}{r}\mathcal{M}^{(20)}(t-r^*) + O(1/r^2) \\
\mathcal{M}_0^{(2)} &= r\mathcal{M}_0^{(20)}(t-r^*) + O(r^0) \\
\mathcal{M}_1^{(2)} &= r\mathcal{M}_1^{(20)}(t-r^*) + \mathcal{M}_1^{(21)}(t-r^*) + O(1/r) \\
\mathcal{M}_2^{(2)} &= \frac{1}{r}\mathcal{M}_2^{(20)}(t-r^*) + O(1/r^3)
\end{aligned} \tag{54}$$

and if we compute them in terms of F_ψ,F_Q, we finally get

$$\begin{aligned}
\frac{\partial \mathcal{M}^{(20)}(t-r^*)}{\partial t} &= \frac{1}{2}F_\chi^{(20)}(t-r^*) - \frac{1}{28}F_\psi^{(v)}(t-r^*)F_\psi^{(iii)}(t-r^*) - \frac{1}{28}F_\psi^{(iv)}(t-r^*)^2 - \frac{3}{14}F_Q^{(iv)}(t-r^*)^2 \\
\mathcal{M}_0^{(20)}(t-r^*) &= \frac{1}{2}F_\chi^{(20)}(t-r^*) - \frac{1}{14}F_\psi^{(v)}(t-r^*)F_\psi^{(iii)}(t-r^*) - \frac{3}{14}F_Q^{(iv)}(t-r^*)^2 \\
\mathcal{M}_1^{(20)}(t-r^*) &= \frac{1}{2}F_\chi^{(20)}(t-r^*) - \frac{1}{14}F_\psi^{(v)}(t-r^*)F_\psi^{(iii)}(t-r^*) - \frac{3}{14}F_Q^{(iv)}(t-r^*)^2 \\
\frac{\partial^2 \mathcal{M}_1^{(21)}(t-r^*)}{\partial t^2} &= \frac{1}{2}F_\chi^{(21)(ii)}(t-r^*) + \frac{3}{14}MF_\psi^{(v)}(t-r^*)^2 - \frac{12}{7}F_\psi^{(iv)}(t-r^*)F_\psi^{(v)}(t-r^*) \\
&\quad - \frac{3}{14}F_\psi^{(ii)}(t-r^*)F_\psi^{(vii)}(t-r^*) - \frac{9}{14}F_\psi^{(vi)}(t-r^*)F_\psi^{(iii)}(t-r^*) \\
&\quad - \frac{69}{14}F_Q^{(v)}(t-r^*)F_Q^{(iv)}(t-r^*) + \frac{9}{14}F_Q^{(vi)}(t-r^*)F_Q^{(iv)}(t-r^*) \\
&\quad + \frac{3}{28}MF_\psi^{(v)}(t-r^*)F_\psi^{(v)}(t-r^*) - \frac{3}{28}MF_\psi^{(iii)}(t-r^*)F_\psi^{(vii)}(t-r^*)
\end{aligned}$$

$$-\frac{15}{14}F_Q^{(vi)}(t-r^*)F_Q^{(iii)}(t-r^*)+\frac{9}{14}MF_Q^{(v)}(t-r^*)^2$$

$$\frac{\partial \mathcal{M}_2^{(20)}(t-r^*)}{\partial t} = F_Q^{(20)}(t-r^*) - \frac{1}{14}F_Q^{(v)}(t-r^*)F_\psi^{(iii)}(t-r^*) - \frac{1}{14}F_Q^{(iii)}(t-r^*)F_\psi^{(v)}(t-r^*) \tag{55}$$

We can now compute the gravitational waveforms by reading off the appropriate metric components. For first order we find,

$$\begin{aligned}
\frac{\partial G^{(1)}(r,t)}{\partial t} &= \frac{1}{r}F_\psi^{(iv)}(t-r^*)+O(1/r^2)\\
\frac{\partial K^{(1)}(r,t)}{\partial t} &= \frac{3}{r}F_\psi^{(iv)}(t-r^*)+O(1/r^2)\\
\frac{\partial h_{2o}^{(1)}(r,t)}{\partial t} &= -2rF_Q^{(iv)}(t-r^*)+O(r^0)
\end{aligned} \tag{56}$$

and the second order components are

$$\begin{aligned}
\frac{\partial G^{(2)}(r,t)}{\partial t} &= \left\{ F_\chi^{20}(t-r^*)+\frac{1}{7}\frac{\partial}{\partial t}\left[F_\psi^{(iv)}(t-r^*)F_\psi^{(iii)}(t-r^*)\right]-\frac{3}{7}F_Q^{(iv)}(t-r^*)^2\right\}\frac{1}{r}+O(1/r^2)\\
\frac{\partial K^{(2)}(r,t)}{\partial t} &= 3\frac{\partial G^{(2)}(r,t)}{\partial t}+O(1/r^2)\\
\frac{\partial h_{2o}^{(2)}(r,t)}{\partial t} &= -2r\left\{F_Q^{20}(t-r^*)+\frac{1}{7}\frac{\partial^2}{\partial t^2}\left[F_Q^{(iii)}(t-r^*)F_\psi^{(iii)}(t-r^*)\right]\right.\\
&\left.\quad+\frac{1}{7}F_Q^{(iv)}(t-r^*)F_\psi^{(iv)}(t-r^*)\right\}+O(r^0)
\end{aligned} \tag{57}$$

and in terms of first and second order Zerilli functions, we get

$$\begin{aligned}
\frac{\partial G^{(1)}(r,t)}{\partial t} &= \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial t}+O(1/r^2)\\
\frac{\partial K^{(1)}(r,t)}{\partial t} &= 3\frac{\partial G^{(1)}(r,t)}{\partial t}\\
\frac{\partial h_{2o}^{(1)}(r,t)}{\partial t} &= -2r\frac{\partial Q^{(1)}(r,t)}{\partial t}+O(r^0)
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\frac{\partial G^{(2)}(r,t)}{\partial t} &= \left\{\chi^{(2)}+\frac{1}{7}\frac{\partial}{\partial t}\left[\frac{\partial \psi^{(1)}}{\partial t}\psi^{(1)}\right]-\frac{3}{7}\left[\frac{\partial Q^{(1)}}{\partial t}\right]^2\right\}\frac{1}{r}+O(1/r^2)\\
\frac{\partial K^{(2)}(r,t)}{\partial t} &= 3\frac{\partial G^{(2)}(r,t)}{\partial t}+O(1/r^2)\\
\frac{\partial h_{2o}^{(2)}(r,t)}{\partial t} &= -2r\left\{\Theta^{(2)}+\frac{1}{7}\frac{\partial^2}{\partial t^2}\left[\psi^{(1)}Q^{(1)}\right]+\frac{1}{7}\frac{\partial}{\partial t}\psi^{(1)}\frac{\partial}{\partial t}Q^{(1)}\right\}+O(r^0)
\end{aligned} \tag{59}$$

It is straightforward to compute the radiated energy, since we are in an explicitly asymptotically flat gauge. One can therefore simply apply the formulae stemming from Landau and Lifshitz [6,7],

$$\frac{d\text{Power}}{d\Omega} = \frac{1}{16\pi r^2}\left[\left(\frac{\partial}{\partial t}h_{\theta\phi}\right)^2+\frac{1}{4}\left(\frac{\partial}{\partial t}h_{\theta\theta}-\frac{1}{\sin^2\theta}\frac{\partial}{\partial t}h_{\phi\phi}\right)^2\right]. \tag{60}$$

For the perturbations we are considering,

$$\begin{aligned}
h_{\theta\theta} &= r^2\left\{[\epsilon\widetilde{K}^{(1)}(r,t)+\epsilon^2\widetilde{K}^{(2)}(r,t)]P_2(\theta)\right.\\
&\left.\quad+[\epsilon\widetilde{G}^{(1)}(r,t)+\epsilon^2\widetilde{G}^{(2)}(r,t)]\partial^2P_2(\theta)/\partial\theta^2)\right\}
\end{aligned} \tag{61}$$

$$h_{\phi\phi} = r^2 \left\{ \sin^2\theta[\epsilon\widetilde{K}^{(1)}(r,t) + \epsilon^2\widetilde{K}^{(2)}(r,t)]P_2(\theta) \right.$$
$$\left. + \sin(\theta)\cos(\theta)[\epsilon\widetilde{G}^{(1)}(r,t) + \epsilon^2\widetilde{G}^{(2)}(r,t)]\partial P_2(\theta)/\partial\theta) \right\} \tag{62}$$
$$h_{\theta\phi} = \frac{1}{2}\left[\epsilon\widetilde{h}_{2o}^{(1)}(r,t) + \epsilon^2\widetilde{h}_{2o}^{(2)}(r,t)\right]\left[\cos(\theta)\partial P_2(\theta)/\partial\theta - \sin(\theta)\partial^2 P_2(\theta)/\partial\theta^2\right] \tag{63}$$

and substituting, we get,

$$\text{Power} = \frac{3}{10}\left[\epsilon\frac{\partial\psi^{(1)}}{\partial t} + \epsilon^2\left(\chi^{(2)} + \frac{1}{7}\frac{\partial}{\partial t}\left(\psi^{(1)}\frac{\partial\psi^{(1)}}{\partial t}\right) - \frac{3}{7}\left(\frac{\partial Q^{(1)}}{\partial t}\right)^2\right)\right]^2$$
$$+ \frac{36}{35}\left[\epsilon\frac{\partial Q^{(1)}}{\partial t} + \epsilon^2\left(\Theta^{(2)} + \frac{1}{7}\left(\frac{\partial^2 Q^{(1)}\psi^{(1)}}{\partial t^2} + \frac{\partial Q^{(1)}}{\partial t}\frac{\partial\psi^{(1)}}{\partial t}\right)\right)\right]^2. \tag{64}$$

As can be seen, the energy has two quadratic contributions, one coming from the $h_{\theta\phi}$ component of the metric and one from the diagonal elements of it. They correspond to the "×" and "+" polarization modes of the gravitational field and therefore we can consider the expressions in the square brackets as the"waveforms" of the gravitational waves emitted. Contrary to the first order case, in which the waveform is directly given by the Zerilli (or Regge–Wheeler) function, for second order corrections the relationship is more involved.

V. INITIAL DATA

Initial data for solutions of Einstein equations is usually given in terms of the initial value of the three-metric and the initial value of the extrinsic curvature of the initial Cauchy surface. For the case of initial data for colliding black holes, the popular families of initial data of Bowen and York [8] and punctures [9] naturally come cast in a gauge in which[1] $H_0 = 0$, $H_1 = 0$, $h_0 = 0$ and $h_{0o} = 0$. We will therefore present the appropriate formulae that would allow us, given a metric and extrinsic curvature cast in such a gauge, to provide the initial data for the first and second order Zerilli functions and their time derivatives.

The extrinsic curvature K_{ab} of a spatial slice is defined by

$$K_{ab} \;=\; n_{(c;d)}\;(\delta^c{}_a + n^c n_a)\left(\delta^d{}_b + n^d n_b\right) \tag{65}$$

where n_a is the unit normal to the Cauchy surface where the data are specified. One can relate the extrinsic curvature to the time derivative of the three-metric through,

$$ds^2 = g_{ij}dx^i dx^j - N^2(dt)^2$$
$$\frac{\partial g_{ij}}{\partial t} = -2NK_{ij}. \tag{66}$$

That is, if through some procedure we are given the extrinsic curvature and the three metric, we can compute the time derivative of the three metric. Re-expressing this in terms of components of the tensor spherical harmonic decomposition due to Regge and Wheeler we get,

$$\begin{array}{rclrcl} \partial h_1^{(1)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(1)}_{(h_1)} & \partial H_2^{(1)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(1)}_{(H_2)} \\ \partial K^{(1)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(1)}_{(K)} & \partial G^{(1)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(1)}_{(G)} \\ \partial h_{1o}^{(1)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(1)}_{(h_{1o})} & \partial h_{2o}^{(1)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(1)}_{(h_{2o})} \end{array} \tag{67}$$

for first order components. $K^{(1)}_{(xx)}$ are spherical tensor harmonic components of the first order extrinsic curvature tensor, written in the Regge–Wheeler notation that we used in formulae (1-10). For second order we get,

$$\begin{array}{rclrcl} \partial h_1^{(2)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(2)}_{(h_1)} & \partial H_2^{(2)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(2)}_{(H_2)} \\ \partial K^{(2)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(2)}_{(K)} & \partial G^{(2)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(2)}_{(G)} \\ \partial h_{1o}^{(2)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(2)}_{(h_{1o})} & \partial h_{2o}^{(2)}/\partial t &=& -2\sqrt{1-2M/r}\,K^{(2)}_{(h_{2o})} \end{array} \tag{68}$$

---

[1] This happens to coincide exactly with the same gauge condition of the asymptotically flat gauge.

Here $K^{(2)}_{(xx)}$ are spherical harmonic components of the second order extrinsic curvature tensor. Note that the expressions are similar to the first order ones only in the gauge we are considering. In other gauges (in which the perturbations of lapse and shift may not vanish), there will be extra terms, quadratic in first order elements.

A case in which the formalism just developed will be of use (and we will discuss in the forthcoming publication) is that of the collision of counterrotating Bowen–York holes (the cosmic screw case). In such a case, to first order the metric has even portions only (similar to those in the Misner [10] initial data) and the extrinsic curvature odd portions. So putting together the above formulae, for this case one gets,

$$
\begin{aligned}
\chi^{(2)} &= -\frac{2}{7}\frac{\sqrt{r-2M}}{(2r+3M)\sqrt{r}}\left[7\,r^2K^{(2)}_{(K)} - 7\,r^2K^{(2)}_{(G)} - 7\,rK^{(2)}_{(h_1)} + 2\,\left(K^{(1)}\right)^2 r\sqrt{-\frac{r}{-r+2\,M}} + \right.\\
&\left. + \,3\,\left(K^{(1)}\right)^2 M\sqrt{-\frac{r}{-r+2\,M}} + 14\,K^{(2)}_{(h_1)}M + 21\,rK^{(2)}_{(G)}M\right]\\
\frac{\partial\chi^{(2)}}{\partial t} &= -\frac{1}{14}\frac{r-2M}{r^4(2r+3M)}\left[140\,Mrh_1^{(2)} + 84\,M^2r^2G_r^{(2)} + r^5\left(K_r^{(1)}\right)^2 - 15\,r^3\left(K^{(1)}\right)^2 - 2\,Mr^4\left(K_r^{(1)}\right)^2 + \right.\\
&+ \,2\,Mr^2\left(K^{(1)}\right)^2 + 2\,r^4K^{(1)}K_r^{(1)} + 36\,r^3\left(K^{(1)}_{(h_{1o})}\right)^2 + 96\,r\left(K^{(1)}_{(h_{2o})}\right)^2 + 276\,M\left(K^{(1)}_{(h_{2o})}\right)^2 - \\
&- \,144\,M^2r\left(K^{(1)}_{(h_{1o})}\right)^2 + 72\,r^2K^{(1)}_{(h_{1o})}K^{(1)}_{(h_{2o})} + 12\,r^2K^{(1)}_{(h_{2o})}K^{(1)}_{(h_{2o})r} - 14\,Mr^3K_r^{(2)} + 28\,r^3K^{(2)} - \\
&- \,28\,r^5G_{rr}^{(2)} - 28\,r^4G_r^{(2)} - 144\,MrK^{(1)}_{(h_{1o})}K^{(1)}_{(h_{2o})} - 24\,MrK^{(1)}_{(h_{2o})}K^{(1)}_{(h_{2o})r} + \\
&+ \,84\,M^2r^3G_{rr}^{(2)} + 14\,Mr^4G_{rr}^{(2)} - 42\,Mr^3G_r^{(2)} - 168\,M^2rh^{(2)}_{1\,r} - 28\,Mr^2h^{(2)}_{1\,r} + 56\,r^3h^{(2)}_{1\,r} + \\
&\left. + \,168\,M^2h_1^{(2)} - 56\,r^2h_1^{(2)} - 28\,r^3H_2^{(2)} - 28\,Mr^2H_2^{(2)}\right]\\
\Theta^{(2)} &= -\frac{(-r+2\,M)\left(-2\,h^{(2)}_{2o} + rh^{(2)}_{2o\,r} + 2\,rh^{(2)}_{1o}\right)}{2\,r^3}\\
\frac{\partial\Theta^{(2)}}{\partial t} &= \frac{1}{7}\frac{\sqrt{r-2M}}{r^3\sqrt{r}}\left[r^2K^{(1)}_{(h_{1o})}K^{(1)} - 14\,r^2K^{(2)}_{(h_{1o})} - 7\,r^2K^{(2)}_{(h_{2o})r} - 35\,K^{(2)}_{(h_{2o})}M + 14\,rK^{(2)}_{(h_{2o})} + \right.\\
&+ \,6\,rK_r^{(1)}K^{(1)}_{(h_{2o})}M + 8\,rK^{(1)}K^{(1)}_{(h_{2o})r}M + 14\,rK^{(2)}_{(h_{2o})r}M + 28\,rK^{(2)}_{(h_{1o})}M - 4\,r^2K^{(1)}K^{(1)}_{(h_{2o})r} - \\
&\left. - \,3\,r^2K_r^{(1)}K^{(1)}_{(h_{2o})} + 8\,rK^{(1)}K^{(1)}_{(h_{2o})} - 20\,K^{(1)}K^{(1)}_{(h_{2o})}M - 2\,rK^{(1)}_{(h_{1o})}K^{(1)}M\right]
\end{aligned}
\tag{69}
$$

VI. CONCLUSIONS

We have written explicitly the second order perturbative equations for axisymmetric $\ell = 2$ perturbations including odd parity modes. We set up explicitly the evolution equations and the formulae for generating initial data starting from a metric and an extrinsic curvature. We also performed an asymptotic analysis to obtain formulas for radiated energies and waveforms. These formulae will be useful for considering perturbatively the evolution of spacetimes arising from the collision of nearby black holes. They might also be useful for other comparison with nonlinear evolutions, as the ones considered in [11].

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